

Damage spreading in the Bak-Sneppen model: Sensitivity to the initial conditions and equilibration dynamics

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The short-time and long-time dynamics of the Bak-Sneppen model of biological evolution are investigated using the damage spreading technique. By defining a proper Hamming distance measure, we are able to make it exhibit an initial power-law growth which, for finite size systems, is followed by a decay towards equilibrium. In this sense, the dynamics of self-organized critical states is shown to be similar to the one observed at the usual critical point of continuous phase-transitions and at the onset of chaos of non-linear low-dimensional dynamical maps. The transient, pre-asymptotic and asymptotic exponential relaxation of the Hamming distance between two initially uncorrelated equilibrium configurations is also shown to be fit within a single mathematical framework.

Keywords: Damage spreading, critical dynamics, Bak-Sneppen model

I. INTRODUCTION

The concept of self-organized criticality was introduced to describe the tendency of large dynamical systems to organize themselves in an out of equilibrium state exhibiting spatio-temporal complexity [1]. Such complexity is reflected by the presence of correlations between events separated in space and time over a wide range of length and time scales [2,3]. The emergence of spatio-temporal complexity in dynamical systems driven by rules based on extremal principles is in the origin of the widespread occurrence in nature of fractal structures [4], noise with $1/f$ power spectrum and punctuated equilibrium [5].

The simplest mathematical model of a driven dynamical system, which exhibits self-organized criticality, was introduced by Bak and Sneppen as a toy model of biological evolution of an ecology of interacting species [6]. In this model random numbers f_i (fitness), uniformly distributed on the interval $[0,1]$, are assigned to the sites of a d -dimensional lattice. At each time step the site with the smaller random number is located and new random numbers from a uniform distribution are assigned to that site and also to its first neighbors.

The above model system achieves a statistically stationary state in which the density of random numbers in the system vanishes for $f < f_c$ and is uniform above f_c , with $f_c = 0.667$ [6] in a chain and $f_c = 0.38$ in a square lattice geometry [5]. Once the stationary state is achieved the system exhibits punctuated equilibrium as real biology [7], characterized by intermittent co-evolutionary avalanches of all sizes. The complexity of this stationary regime can be revealed by the show up of spatio-temporal power-law distribution of events. For example, the distribution $C(x)$ of the distance x between two subsequent extremal sites scales as $C(x) \propto x^{-\pi}$, with $\pi = 3.23$ in one dimension [5]. Further the temporal long-range correlations can be also observed in the distribution of first return times $P(t)$ which scales as $t^{-\tau_{first}}$, with $\tau_{first} = 1.58$ in $1d$ and $\tau_{first} = 1.28$ in $2d$ [5].

The propagation of local perturbations in the self-organized state of Bak-Sneppen model was recently investigated using the damage spreading algorithm [8–12]. By measuring how the difference between two initially close configurations evolves in time under the same noise, it was shown that an initial power-law divergence sets up before the distance saturates in a finite size-dependent plateau. This slow short-time power-law dynamics is characteristics of systems poised at critical states such as at the critical point of second-order phase transitions [13] and the onset of chaos in non-linear dynamical systems [14–22].

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However, usual critical phenomena exhibit two dynamical regimes characterized by distinct critical exponents. The first one is related to a short-time slow dynamics, which governs the power-law increase of the order parameter when its value in an initial non-equilibrium configuration is small and non-zero [13]. In finite systems this power-law increase persists up to a characteristic time, after which the order parameter slowly decays towards its equilibrium value.

The same dynamical regimes have been identified in the onset of chaos of low-dimensional non-linear dynamical maps. The distance between two nearby orbits was shown to diverge following a power-law whose exponent is directly related to geometric exponents characterizing the extremal sets of the critical dynamical attractor [14–22]. On the other hand, the long-time relaxation towards the dynamical attractor is governed by a new exponent, which seems to be related to the fractal dimension of the support of the dynamical attractor in phase-space [23].

Damage spreading studies of critical phenomena have also reported two dynamical regimes with distinct and independent exponents. Therefore, it is natural to expect this to be the general trend of systems poised at criticality as well as systems exhibiting self-organized criticality such as the Bak-Sneppen model. The fact that previous damage spreading works have not identified the long-time relaxation towards equilibrium in the BS model just reflects the insensitivity of the proposed measure to capture the long-time dynamics.

In this work we introduce a new Hamming distance measure that is sensitive to the long-time dynamics of the Bak-Sneppen model. Following its temporal evolution we are going to show that, besides displaying a power-law short-time dynamics with the same exponent previously reported, it also exhibits a long-time relaxation dynamics. We will report the scaling properties of this regime and discuss its possible relation with the geometric properties of the dynamical attractor in phase space.

II. MODEL SYSTEM AND NUMERICAL SIMULATION

In our simulations we implement the Bak-Sneppen algorithm in linear chains with periodic boundary conditions. Random numbers f_i (fitness) are initially assigned to all sites from a uniform distribution in the interval $[0, 1]$. The system evolution is based on the standard re-assignment of the extremal site, i.e., the site with the minimum fitness and of its first neighbors. In our simulations we worked with chains of up to $N = 1000$ sites and left $t_{trans} = 20N$ collective time steps for the system to achieve the statistically stationary state. Our collective time unit corresponds to N elementary time steps so that each site is going to be updated only once in average during a collective time step.

After achieving the statistically stationary state we produce a copy of the system's configuration. A small damage is introduced in the copy by interchanging the position of the site with the minimum fitness with a randomly chosen one. After this, we follow the temporal evolution of both configurations using always the same set of random numbers to update both replicas. After a characteristic time τ , both configurations will be composed by the same sequence of random numbers just shifted by a random distance. Once periodic boundary conditions are used, these two configurations are indeed indistinguishable and should be considered as identical. In figure 1 we report the finite size scaling of the average time needed for two initially random configurations to become identical when updated following the Bak-Sneppen dynamics and under the same set of random numbers. This characteristic time, measured in units of collective time steps, scales as $\tau \propto N^\phi$, with $\phi = 1.46(4)$. We would like to stress that, within the error bars, this is the same scaling exponent obtained by the size dependence of the average self-organization time to reach the critical attractor [$D - 1 = 1.43(1)$] [5] and is independent of the initial state of the system.

In order to accomplish our task, we introduce a new Hamming distance $D(t)$ defined at each time t as the smallest among the N possible values of $D_j(t)$ ($j = 1, 2, \dots, N$) defined as

$$D_j(t) = \left\langle \frac{1}{N} \sum_{i=1}^N |f_i^1 - f_{i+j}^2| \right\rangle . \quad (1)$$

where $\langle \dots \rangle$ represents configurational average over distinct runs. Those runs at which the Hamming distance vanishes are not considered in the averaging process and therefore our simulations are limited in size by the characteristic time τ . Notice that $D_0(t)$ corresponds to the measure used in previous damage spreading studies of Bak-Sneppen model [8–12]. After the initial power-law growth $D_0(t) \propto t^\alpha$ with $\alpha = 0.32$, it saturates at a constant value which can be shown to be $1/3$ of the width of the fitness distribution in equilibrium [12]. This saturation is due to the fact that by just measuring $D_0(t)$ one cannot identify that both replicas have converged to the same random sequence configuration.

The procedure proposed here is to measure the Hamming distance as the minimum distance between the replicas taking in account all possible shifts between them. In respect to the short-time dynamics it is equivalent to measuring $D_0(t)$ itself and shall give the same power-law exponent for the initial Hamming distance growth. However, after achieving a maximum, it must decrease to vanishing small values as both replicas converge to the same sequence, capturing then the long-time dynamics.

In figure 2 we show our results for $D(t)$ for chains up to $N = 1000$ sites. As larger chain sizes are considered the initial power-law regime extends for longer periods. The dashed line corresponds to the power law $D(t) \propto t^\alpha$ with $\alpha = 0.32$ in full agreement with the value previously reported in the literature [8]. After a size dependent characteristic time, it reaches a maximum and starts to monotonically decrease. However, much longer runs than these presently reported would be needed to precisely estimate the scaling behavior of the Hamming distance decay.

In order to investigate in detail the long-time dynamical regime, it is computationally more efficient to use a slightly different approach. We start with two uncorrelated equilibrium configurations and follow the time evolution of the distance between them when the same set of random numbers is used. In this way, the initial growth is absent and the relaxation regime can be observed using shorter runs. Usually the critical power-law relaxation dynamics takes place after some transient time. This typical behavior means that the Hamming distance at criticality satisfies the non-linear differential equation

$$\frac{dD(t)}{dt} = -\lambda D^q \quad (2)$$

whose solution is

$$D(t) = D(0)/[1 + \lambda(q-1)t]^{1/(q-1)} \quad (3)$$

and reproduces the above trend. In numerical simulations, the critical relaxation dynamics is always followed by an exponential relaxation due to finite size effects. A pure exponential relaxation implies that $D(t)$ must asymptotically obey a linear differential equation. Therefore in order to consider simultaneously the critical and exponential relaxation we consider $D(t)$ to obey a more general non-linear differential equation such as

$$\frac{dD(t)}{dt} = -(\lambda - \mu)D^q - \mu D \quad (4)$$

In the above equation, μ represents the finite size correction and vanishes in the thermodynamic limit. The general solution of the above differential equation is in the form

$$D(t) = D(0)/[1 - (\gamma/\mu) + (\gamma/\mu)e^{(\lambda-\mu)t}]^{1/(q-1)} \quad (5)$$

where $\gamma = \mu - (\mu - \lambda)/D_0^{(1-q)}$. It is worth to mention that similar equations have also been used to describe experimental data on the re-association in folded proteins [24], quantitative linguistics [25] and fluxes of cosmic rays [26] within the framework of nonextensive thermostatics [27,28]. Indeed, the above form is expected to generally describe the sensitivity to initial conditions of systems at the vicinity of critical states.

In Figure 3 we report our results for $D(t)$ starting with two uncorrelated equilibrium configurations for several chain sizes. The average initial distance equals to $D(0) = 0.111$ which corresponds to $1/3$ of the width of the interval $[f_c, 1]$ where $f_c = 0.667$ for the $1d$ Bak-Sneppen model. The coefficient μ determines characteristic exponential decay which sets up in the long-time regime as can be seen in the inset. The solid lines correspond to best fits to the form of Eq. (5). We recall that the fitting parameters govern the typical behavior at distinct time regimes. In the table we summarize the parameters obtained. Notice that μ decreases monotonically when chain size is increased, once the exponential relaxation is postponed for large chains. Also γ decreases with N so that the transient period diverges in the thermodynamic limit. However, $\mu \ll \gamma$ and therefore the pre-asymptotic decay can be observed just at intermediate times satisfying $1/\gamma \ll t \ll 1/\mu$.

In figure 4 we show the size dependence of μ and γ . Once $1/\mu$ determines the time scale needed to finite size effects become relevant, it shall present the same scaling behavior as the average time τ for both replicas to become identical. We found $1/\mu \propto N^{1.49(4)}$, which is consistent with the above conjecture. On the other hand, $1/\gamma$ defines the time scale when the crossover between the transient and the pre-asymptotic regime takes place. It is governed by a distinct exponent $1/\gamma \propto N^\kappa$, with $\kappa = 1.11(3)$ and is directly related to the average time needed to update all active sites in the initial avalanche [5].

For the system sizes simulated, the intermediate regime satisfying the condition $1/\gamma \ll t \ll 1/\mu$ is achieved only in a narrow time interval which grows slowly as the chain size is increased. This feature introduces a large uncertainty in the estimates of the non-linear exponent q , particularly the ones from small chain data. The best fitting values for q were found to be much larger than one, specially the more confident estimate from chain size $N = 1000$. This may indicate a very slow logarithmic pre-asymptotic decay of the Hamming distance.

At this point it is worth to explore the similarities between the damage spreading analysis of the dynamical properties of extended critical systems and recent results concerning the sensitivity to initial conditions of low-dimensional dissipative maps poised at criticality [14–22]. More precisely, the numerical analysis of such systems (i.e.,

logistic map [14], logistic-like maps [15], circular maps [18], asymmetric logistic maps [19], single-site map [21], Henon map [22]) has shown that, at critical points such as the chaos threshold, tangent bifurcations etc, the standard type of the sensitivity to the initial conditions given by the sensitivity function

$$\xi(t) \equiv \lim_{\Delta x(0) \rightarrow 0} \frac{\Delta x(t)}{\Delta x(0)} = \exp(\lambda_1 t) \quad (6)$$

(where λ_1 is the standard Lyapunov exponent) can be replaced by a power-law type of sensitivity

$$\xi(t) = [1 + (1 - q)\lambda_q t]^{1/(1-q)}, \quad (7)$$

which indicates a weak sensitivity to the initial conditions. Here, λ_q is the generalized Lyapunov exponent, and satisfies the generalized Pesin equality $K_q = \lambda_q$ if $\lambda_q > 0$ and $K_q = 0$ otherwise, where K_q is the generalized Kolmogorov-Sinai entropy [14].

Now we are in a position to discuss the connection between this scenario and the BS model. Indeed, the BS model belongs to another class of dynamical systems, namely high-dimensional dissipative systems, for which this scenario is also needed to be tested. As a matter of fact, the BS model evolves spontaneously (without a tuning parameter) towards a critical state, whereas a one-dimensional map approaches to a critical point (e.g. chaos threshold point) with a fine tuning of a map parameter. Moreover, in both cases, the standard Lyapunov exponent would be zero. Therefore, the properties of the sensitivity to the initial conditions of the BS model are expected to be similar to those of the low-dimensional dissipative maps at the edge of chaos. Sensitivity to the initial conditions, which is reflected by the sensitivity function for the low-dimensional dissipative maps, can be conveyed by the damage spreading (and Hamming distance) for the BS model. It is clear from ref. [8] and from our results using a different definition of the Hamming distance that a power-law spread of damage emerges for the BS model in the short-time dynamics regime. Then, one can easily determine the proper q value of the BS model as $q^* \simeq -2.1$.

In what follows we discuss the picture seen from the analysis of long-time dynamics of the BS model. Firstly, we should recall the results of low-dimensional dissipative maps. Recently, it has been shown [23] that, for the long-time dynamics of such dynamical systems, a new class of q values emerges. This new class appeared to be related to the equilibration (or long-time dynamics). It is seen from the results of [23] that, whenever the critical attractor of the dynamical system has a fractal support (or in other words, the critical attractor is not dense), the system (e.g., logistic-like maps) exhibits a power-law long-time relaxation with a q value greater than unity. On the other hand, if the critical attractor of the dynamical system (e.g., circular maps) is dense, then a slow logarithmic decay has been observed. When we look at the BS model, the situation is similar to that of the circular maps in the sense that the attractor in phase space is fully occupied in the interval $[f_c, 1]$, which makes the critical attractor dense. The observed slow (logarithmic) pre-asymptotic dynamics seems to indicate that it shall take place in general system models whose critical dynamical attractor is dense.

III. SUMMARY AND CONCLUSIONS

In summary, we have applied the damage spreading algorithm to study the dynamics of the Bak-Sneppen model for biological evolution which is one of the prototypes models exhibiting self-organized criticality. By defining a proper Hamming distance between two system's configurations we investigated both the short-time and long-time critical dynamics. A power-law short-time dynamics was found in agreement with previous reports [8–12]. In the relaxation regime, the Hamming distance was shown to follow the solution of a generalized non-linear equation, first proposed in the context of nonextensive thermostatics, which includes in a single expression the transient, pre-asymptotic and asymptotic exponential relaxation regimes. We found that the pre-asymptotic regime is very close to a slow logarithmic decay in a close relationship with the relaxation dynamics of non-linear dissipative maps at the onset of chaos with a dense dynamical critical attractor. It would be interesting to investigate the relaxation dynamics of self-organized critical systems with a fractal dynamical attractor in phase-space. In this case, based on previous studies of the critical dynamics of low-dimensional deterministic non-linear maps at the onset of chaos, one would expect a power-law relaxation. This point will be the subject of a future contribution.

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- [1] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).
- [2] Y.-C. Zhang, J. Phys. I (France) **1**, 971 (1991).
- [3] H.C. Fogedby, J. Stat. Phys. **69**, 411 (1992).
- [4] See e.g. B.B. Mandelbrot, *The Fractal Geometry of Nature*, (Freeman, New York 1983).
- [5] M. Paczuski, S. Maslov and P. Bak, Phys. Rev. E **53**, 414 (1996).
- [6] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).
- [7] S.J. Gould and N. Eldredge, *Paleobiology* **3**, 114 (1997); *Nature* **366**, 223 (1993).
- [8] F.A. Tamarit, S.A. Cannas and C. Tsallis, Eur. Phys. J. B **1**, 545 (1998).
- [9] P.M. Gleiser, F.A. Tamarit and S.A. Cannas, *Physica A* **275**, 272 (2000).
- [10] R. Cafiero, A. Valleriani and J.L. Vega, Eur. Phys. J. B **7**, 505 (1999).
- [11] R. Cafiero, A. Valleriani and J.L. Vega, Eur. Phys. J. B **4**, 405 (1998).
- [12] A. Valleriani and J.L. Vega, J. Phys. A: Math. Gen. **32**, 105 (1999).
- [13] H.K. Janssen, B. Schaub and B. Schmittmann, Z. Phys. B **73**, 539 (1989).
- [14] C. Tsallis, A.R. Plastino and W.-M. Zhang, *Chaos Solitons Fractals* **8**, 885 (1997).
- [15] U.M.S. Costa et al., Phys. Rev. E **56**, 245 (1997).
- [16] M.L. Lyra and C. Tsallis, Phys. Rev. Lett. **80**, 53 (1998).
- [17] C.R. da Silva, H.R. da Cruz and M.L. Lyra, *Braz. J. Phys.* **29**, 144 (1999).
- [18] U. Tirnakli, C. Tsallis and M.L. Lyra, Eur. Phys. J. B **11**, 309 (1999).
- [19] U. Tirnakli, C. Tsallis and M.L. Lyra, Phys. Rev. E **65**, 036207 (2002).
- [20] F. Baldovin and A. Robledo, Phys. Rev. E **66**, 045104 (2002) ; *Europhys. Lett.* **60**, 518 (2002).
- [21] U. Tirnakli, *Physica A* **305**, 119 (2002).
- [22] U. Tirnakli, Phys. Rev. E **66** (2002) 066212.
- [23] F.A.B.F. de Moura, U. Tirnakli and M.L. Lyra, Phys. Rev. E **62**, 6361 (2000).
- [24] C. Tsallis, G. Bemsiki and R.S. Mendes, Phys. Lett. A **257**, 93 (1999).
- [25] M.A. Montemurro, *Physica A* **300**, 567 (2001).
- [26] C. Tsallis, J.C. Anjos and E.P. Borges, *Fluxes of cosmic rays: A delicately balanced anomalous thermal equilibrium*, preprint (2002), [e-print: astro-ph/0203258]
- [27] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
- [28] C. Tsallis, in *Nonextensive Statistical Mechanics and its Applications*, edited by S. Abe and Y. Okamoto, Lecture Notes in Physics (Springer, Berlin, 2001).

V. FIGURE AND TABLE CAPTIONS

Figure 1 - The finite size scaling of the average time for two initially uncorrelated random sequences to become identical when updated following the Bak-Sneppen dynamical rules using the same set of random numbers. The solid line corresponds to a power-law fit $\tau \propto N^\phi$ with $\phi = 1.46(4)$. The number of experiments used in calculations for different chain sizes is 100.

Figure 2 - The time evolution of the Hamming distance between an equilibrium configuration and its slightly modified replica. After the initial power-law growth $D(t) \propto t^\alpha$ with $\alpha = 0.32$, it monotonically decreases as both configurations converge to the same sequence of random numbers. The number of experiments used in calculations varies from 100 up to 500 for different chain sizes.

Figure 3 - The time evolution of the Hamming distance between two uncorrelated equilibrium configurations. After the initial transient where the Hamming distance is close to its initial value $D(0) = 0.111$, $D(t)$ exhibits a non-trivial decay followed by an exponential relaxation due to finite size effects. The solid lines correspond to best fits to the form of Eq. (5) that contains the above three regimes. The fitting parameters are reported in the table. The number of experiments used in calculations varies from 100 up to 500 for different chain sizes.

Figure 4 - The finite size dependence of the fitting parameters μ and γ . The solid lines are best fits to power-laws. We found that $1/\mu \propto N^{1.49(4)}$ and therefore it is, within the error bar, proportional to the average time needed to both configurations coincide. On the other hand $1/\gamma \propto N^\kappa$ with $\kappa = 1.11(3)$ has, within the error bar, the same scaling behavior of the average time needed to update all sites in the initial avalanche.

Table Caption - The size dependence of the best fitting parameters. Notice that μ decreases as the chain size N increases, as expected. The large values of q indicate a possible slow logarithmic relaxation in the pre-asymptotic regime as discussed in the text.

Table

N	γ	μ	q
100	0.088	0.012	8.7
200	0.039	0.0043	8.7
500	0.013	0.00094	10.0
1000	0.0062	0.00034	14.1

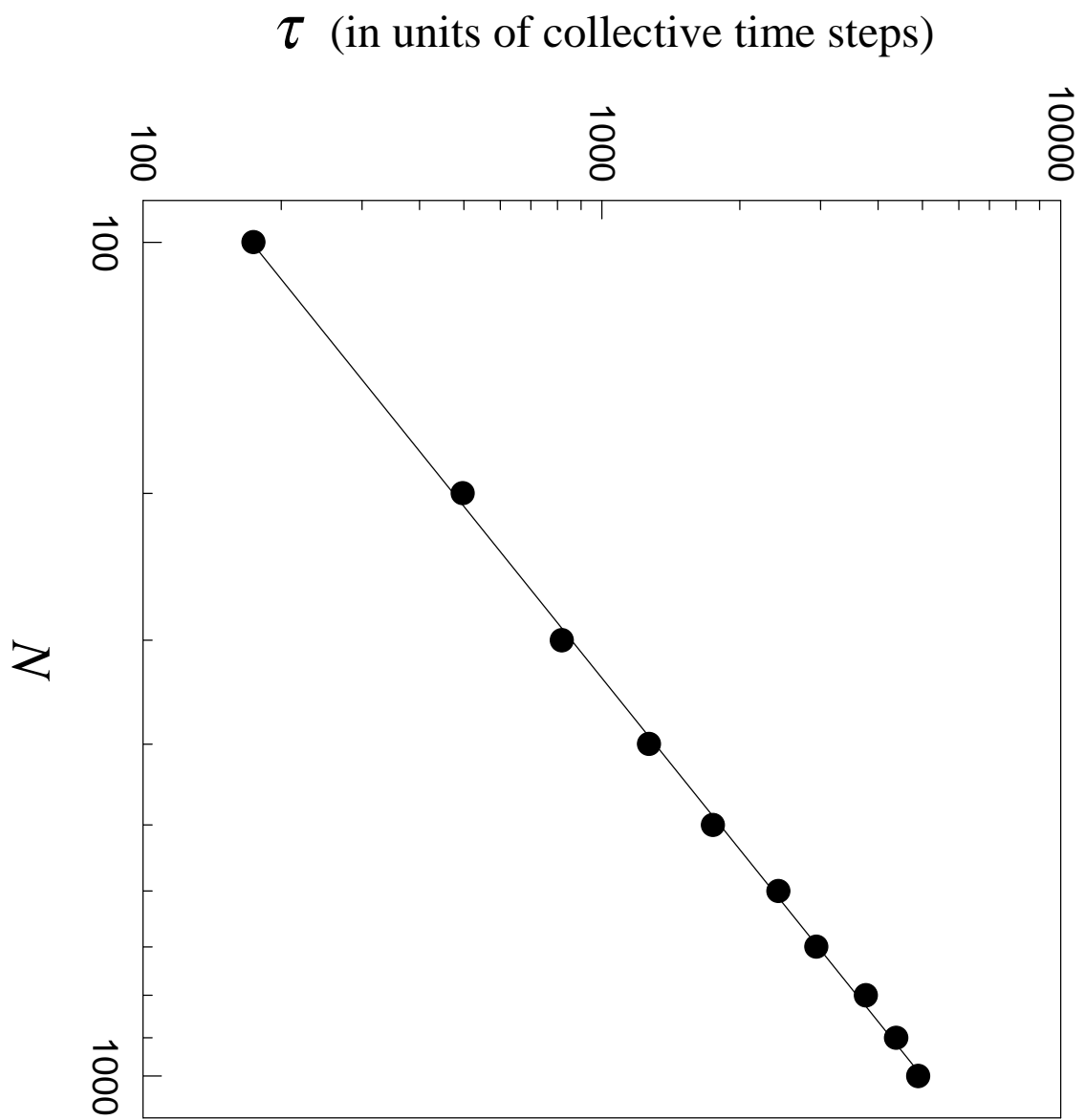


Figure 1

Figure 2

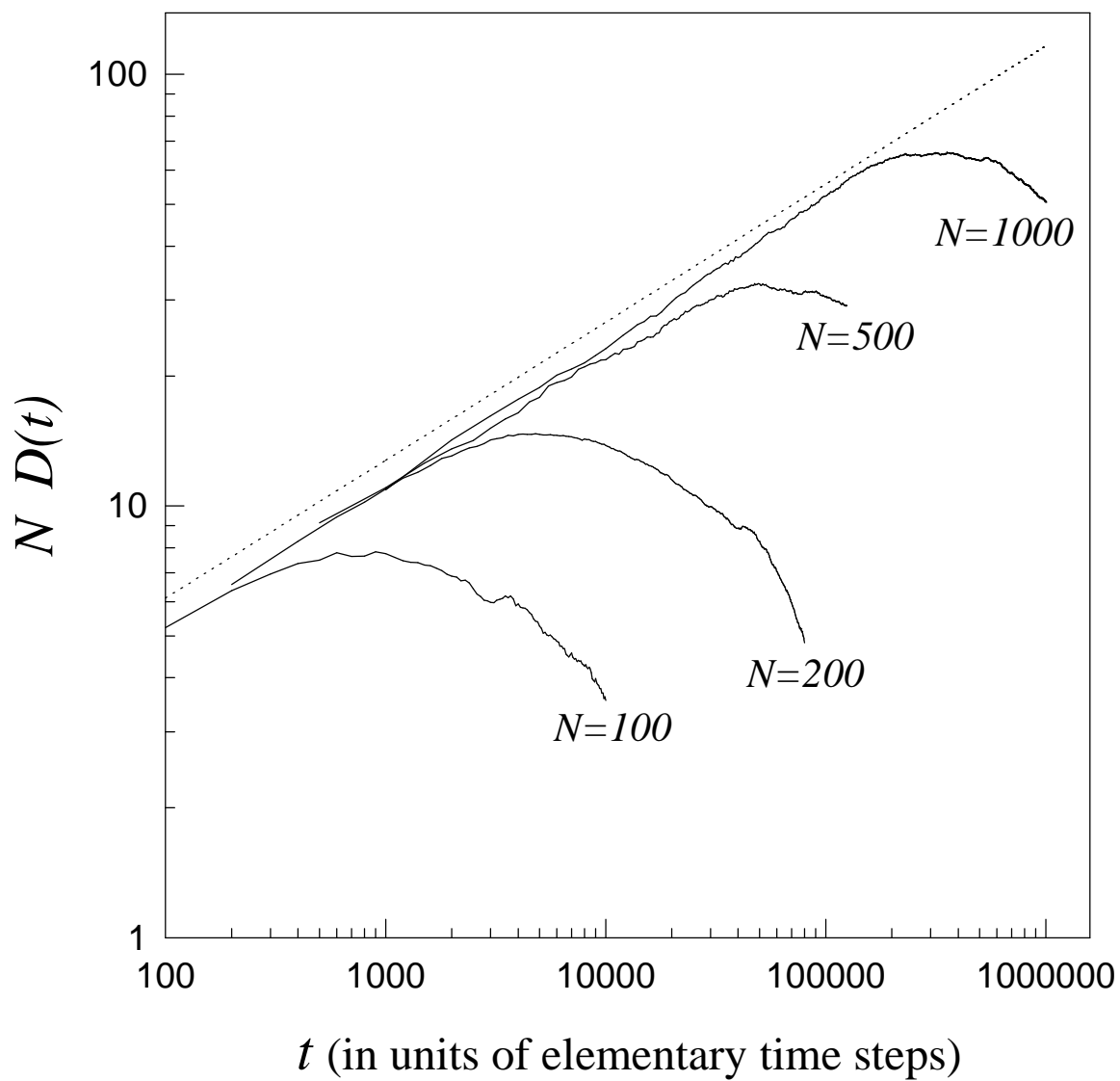


Figure 3

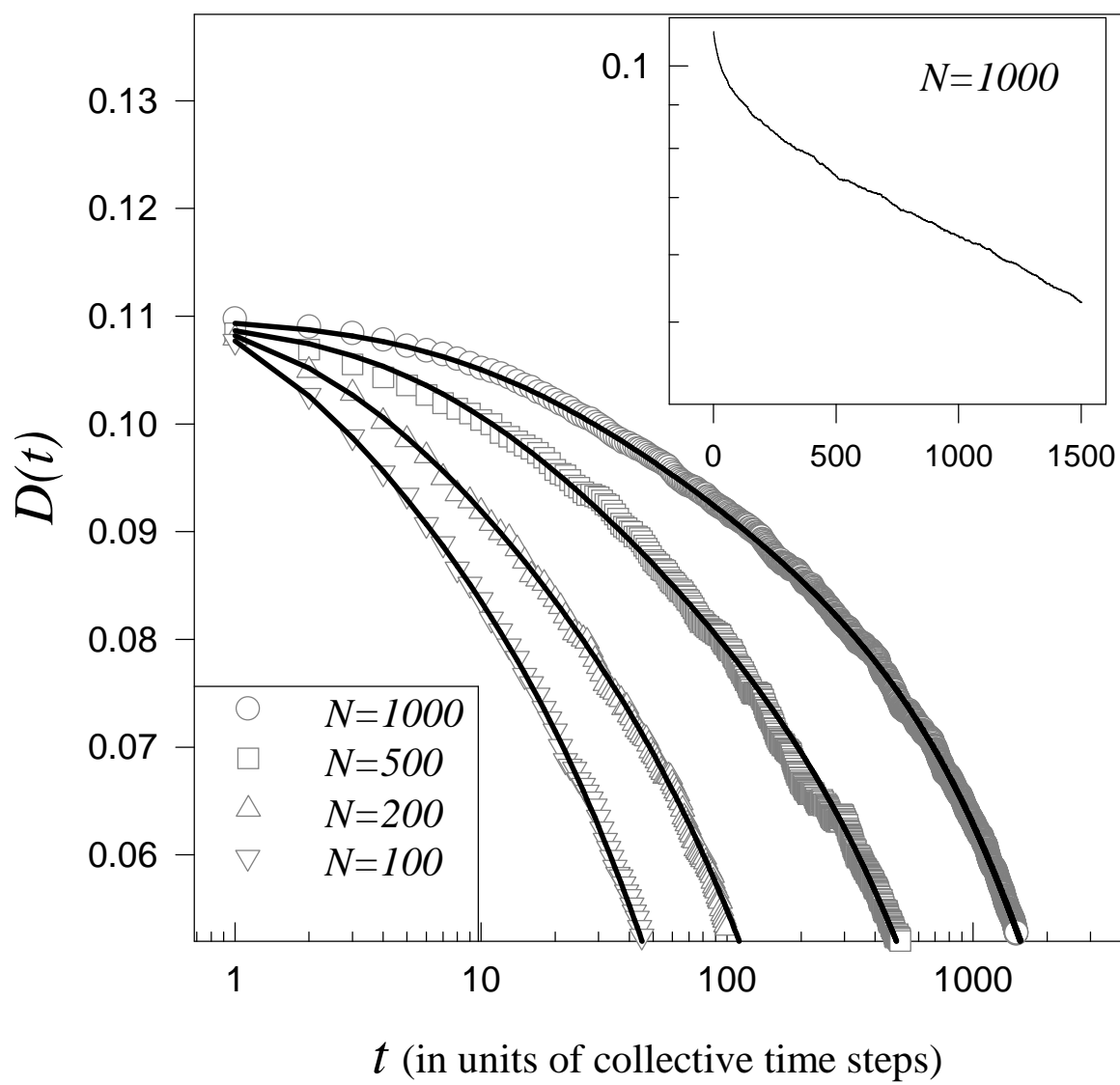


Figure 4

